J. Fluid Mech. (1980), *vol.* 99, *part* 4, *pp.* 801–811
Printed in Great Britain

On the Whitham theory of shock-wave diffraction at concave corners

By L. F. HENDERSON

Department of Mechanical Engineering, University of Sydney,
New South Wales 2006, Australia

(Received 26 October 1979)

The well-known Whitham theory may be applied to shocks diffracting over concave corners, provided that the diffraction results in Mach reflexion. This paper compares the theory with data obtained during experiments with diffracting shocks. If an incident shock is classified as weak or strong in the strict sense defined by von Neumann, then it is found that the Whitham theory accurately determines the Mach number of the Mach stem at the wall for both the weak and the strong cases. The theory also has some further value for strong shocks but not for weak; it is not applicable when the diffraction is regular.

1. Introduction

When a plane shock wave i diffracts over a concave corner of large angle θ_w, then a specular reflexion appears with a reflected shock r (figure 1); this was called regular reflexion RR by von Neumann in 1943 (see von Neumann 1963). If θ_w is reduced sufficiently, the diffraction undergoes transition to an irregular wave system which von Neumann called Mach reflexion MR. There is now both a third shock called the Mach stem s, and a contact discontinuity cd which is caused by the different entropy changes experienced by the gas subject to i and r on one side of cd and s on the other side of it. The exact, perfect gas, theory of RR and MR has been discussed by von Neumann (1943), Eggink (1943), Guderley (1947), Wuest (1948), Bleakney & Taub (1949), Wecken (1949), Kawamura & Saito (1956), Henderson (1964, 1965), Sakurai (1964), Mölder (1971), Henderson & Siegenthaler (1980), and many others.

Von Neumann found it necessary to distinguish between weak and strong incident shocks and gave rigorous definitions of these terms. For a diatomic gas with ratio of specific heats $\gamma \equiv C_p/C_v = \frac{7}{5}$, he found that i was weak if its inverse strength

$$\xi_i \equiv P_0/P_i > 0{\cdot}433,$$

and strong when $\xi_i < 0{\cdot}433$; here P is the pressure, and the subscripts 0, 1 refer to the state of the gas upstream and downstream of i respectively. These inequalities correspond to the shock Mach number M_i ranges of $M_i < 1{\cdot}46$, for the weak case and $M_i > 1{\cdot}46$ for the strong. Experiments by Smith (1945), Kawamura & Saito (1956), Henderson & Lozzi (1975, 1979) and Henderson & Siegenthaler (1980) showed that the von Neumann theory was successful for RR for both weak and strong i. However there was an apparent persistence of RR for a range of θ_w, for which the von Neumann theory had no physically meaningful solutions; some writers have argued that this effect is

0022-1120/80/4504-1680 $02.00

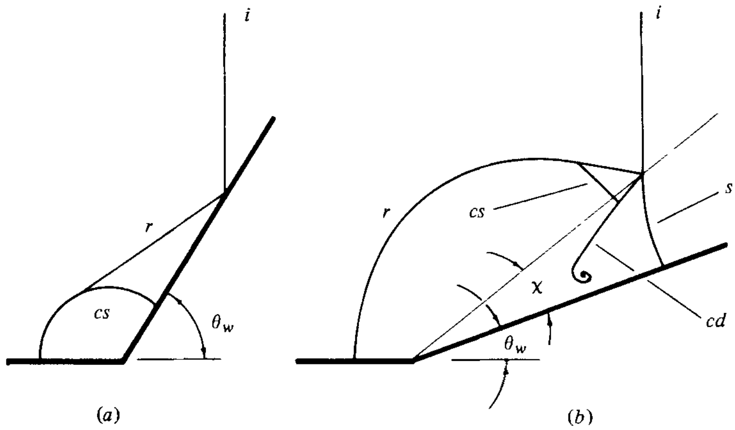

FIGURE 1. Diffracted wave systems on a concave wedge. (*a*) Regular reflexion RR. (*b*) Mach reflexion MR. *i*, incident shock; *r*, reflected shock; *s*, Mach stem; *cs*, corner signal; *cd*, contact discontinuity; θ_w, wedge angle; χ, trajectory path angle.

spurious (Henderson & Lozzi 1975, 1979; Henderson & Siegenthaler 1980). Experiment has also shown the theory to be successful for strong MR but a failure for weak MR. Recently Henderson & Siegenthaler have shown that the failure is due to the theory ignoring the attenuating effect of the corner signal *cs*, which is always able to overtake every part of the shocks *r* and *s* in weak MR, but not always in strong MR.

Whitham (1957, 1959) formulated a new approach to the general problem of a propagating shock (shock dynamics) in a perfect gas and his theory can be applied to shocks diffracting over corners. He considered wave-like disturbances moving along the downstream side of the shock *i*. The theory was straightforward for a convex corner which produced expansive disturbances, but it was more complicated for a concave corner because this produced compressive disturbances which ultimately 'broke' to form a discontinuity. Whitham called it a 'shock–shock' *ss* and it can be thought of as a propagating disturbance which carries a discontinuity in the slope of *i* and in the local ray path. He found that a shock–shock could not properly represent a real shock because of a problem in over-determinism. Nevertheless the theory could represent other parts of a Mach reflexion such as the Mach stem *s* and the trajectory path angle χ of the three-shock confluence, see figure 1. The theory predicted MR for all values of θ_w so that it could never represent regular reflexion except in an asymptotic sense when the length of the Mach stem became vanishingly small with $\theta_w \rightarrow 90°$. The theory ignores the interaction of disturbances with each other and also the effects of any disturbances which originate downstream and subsequently overtake *i*. Whitham concluded that his theory would be of most value for moderately strong MR.

Whitham made a somewhat superficial comparison of his theory with the experimental data obtained by Smith for concave corners. The data was presented by Bleakney & Taub for $M_i = 2{\cdot}42 > 1{\cdot}46$, and there was some systematic discrepancy. Bryson & Gross (1961) made a more rigorous comparison for the exacting problems of shocks diffracting over a cylinder ($M_i = 2{\cdot}82$), a sphere ($M_i = 2{\cdot}85$, $4{\cdot}41$), and several cones ($M_i = 3{\cdot}68$). They measured shock–shock loci and obtained good agreement with the theory. The comparisons which have been made so far concentrate on these loci. In the present paper a more detailed comparison is made using some of the Henderson &

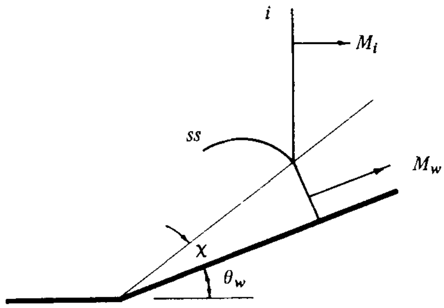

FIGURE 2. Whitham model of shock diffraction. *ss*, shock-shock; M_i, shock Mach number of incident shock i; M_w, shock Mach number of perturbed shock (Mach stem).

Siegenthaler data augmented with additional data obtained by Gray. The Whitham theory is found to be remarkably accurate for predicting the shock Mach number M_w of the Mach stem at the wall, see figure 2, for both weak and strong MR. It is also of some further value for strong MR, but of no other value for weak MR or for RR. To some extent the present work complements the recent paper by Bazhenova, Gvozdeva & Zhilin (1979) on convex corners.

2. The Whitham theory

The Whitham theory of a plane shock wave diffracting over a concave corner in a perfect gas, provides the following expressions for the shock–shock, in present symbolism,

$$\tan\theta_w = \frac{(M_w^2 - M_i^2)^{\frac{1}{2}}(A_i^2 - A_w^2)^{\frac{1}{2}}}{A_w M_w + A_i M_i}, \tag{1}$$

$$\tan\chi = \frac{A_w}{A_i}\left\{\frac{1-(M_i/M_w)^2}{1-(A_w/A_i)^2}\right\}^{\frac{1}{2}}, \tag{2}$$

$$\frac{A_w}{A_i} = \frac{f(M_w)}{f(M_i)}. \tag{3}$$

The function $A = f(M)$ followed from the work of Chester (1954) and Chisnell (1957) on shocks propagating along ducts of slowly varying cross-sectional area A. Some writers refer to the theory as the Chester–Chisnell–Whitham theory CCW. The function $f(M)$ is defined by

$$f(M) = \exp\left\{-\int \frac{M\lambda(M)}{M^2-1}\,dM\right\}, \tag{4}$$

where

$$\lambda(M) \equiv \left[1 + \frac{2}{\gamma+1}\frac{1-\mu^2}{\mu}\right]\left[1 + 2\mu + \frac{1}{M^2}\right], \tag{5}$$

and where

$$\mu^2 \equiv \frac{(\gamma-1)M^2+2}{2\gamma M^2-(\gamma-1)}. \tag{6}$$

Physically μ is the Mach number of the gas normal to, relative to, and downstream of i.

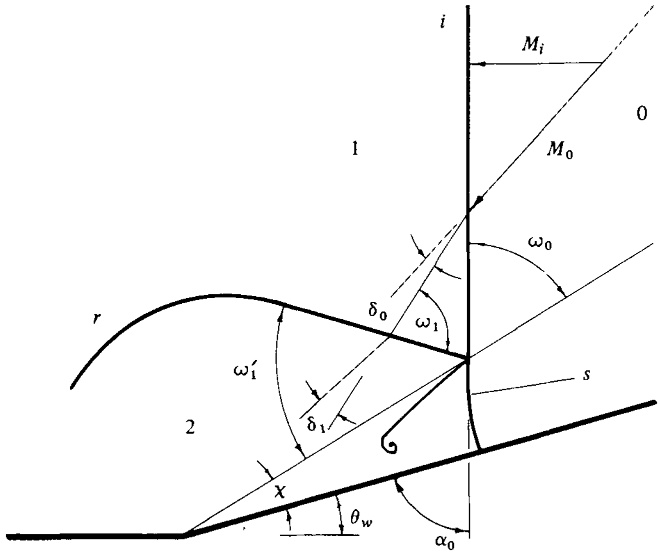

FIGURE 3. Nomenclature in shock confluence co-ordinates. M_0, free-stream Mach number upstream of i; $\delta_{0,1}$, streamline deflexion angles across i and r; ω_0, conventional angle of incidence; α_0, set angle of incidence; ω_1' measurable angle of reflexion; ω_1, true angle of reflexion.

On evaluating the integral one obtained, for $f(M)$,

$$
\begin{aligned}
f(M) \equiv \exp\Bigg\{-\Bigg\{&\ln\frac{M^2-1}{M}+\frac{1}{\gamma}\ln\left(M^2-\frac{\gamma-1}{2\gamma}\right)+\ln\frac{1-\mu}{1+\mu} \\
&+\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}\ln\left[\mu+\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}\right]-\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}\ln\left[\mu-\left(\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}\right] \\
&+\left(\frac{2}{\gamma(\gamma-1)}\right)^{\frac{1}{2}}\ln\left[\left(M^2+\frac{2}{\gamma-1}\right)^{\frac{1}{2}}+\left(M^2-\frac{\gamma-1}{2\gamma}\right)^{\frac{1}{2}}\right] \\
&+\left(\frac{1}{2(\gamma-1)}\right)^{\frac{1}{2}}\tan^{-1}\left\{\frac{[4\gamma-(\gamma-1)^2]M^2-4(\gamma-1)}{4\gamma^{\frac{1}{2}}(\gamma-1)\,[M^2+(2/(\gamma-1))]^{\frac{1}{2}}\,[M^2-(\gamma-1)/2\gamma]^{\frac{1}{2}}}\right\}\Bigg\}\Bigg\}. \qquad (7)
\end{aligned}
$$

So far as I know this last expression has not been published before correctly in terms of the shock Mach number. The expression given by Bryson & Gross has several misprints. Now given $(\gamma, \xi_i, \theta_w)$ one may obtain M_i from Ames (1953),

$$M_i^2 = \frac{\gamma+1}{2\gamma}\xi_i^{-1}+\frac{\gamma-1}{2\gamma}, \qquad (8)$$

and then the shock Mach number M_w of the Mach stem *at the sloping wall* follows from equations (1), (3) and (7), and χ from (2). Further, by geometry, figure 3, one has, for the *set* angle of incidence α_0 of i,

$$\alpha_0 = 90-\theta_w, \qquad (9)$$

and, for the conventional angle of incidence ω_0,

$$\omega_0 = \alpha_0-\chi, \qquad (10)$$

which determine α_0 and ω_0. Then, taking co-ordinates at rest at the shock confluence, the free-stream Mach numbers $M_{0,1}$, upstream, and downstream respectively of i (figure 3) are given by

$$M_0 = M_i/\sin\omega_0, \tag{11}$$

$$M_1^2 = -\frac{2}{\gamma-1}+\left(\frac{2}{\gamma-1}+M_0^2\right)\frac{(\gamma+1)/(\gamma-1)+\xi_i}{(\gamma+1)/(\gamma-1)+\xi_i^{-1}}, \tag{12}$$

and the streamline deflexion angle δ_0 across i is obtained from (Ames 1953)

$$\cot\delta_0 = \left[\frac{\frac{1}{2}(\gamma+1)\,M_0^2}{M_i^2-1}-1\right]\tan\omega_0. \tag{13}$$

The shock Mach number M_r of the reflected shock r can be found from the continuity condition on the pressure at the contact discontinuity in the neighbourhood of the confluence. The condition is

$$\xi_r = \xi_w/\xi_i, \tag{14}$$

where $\xi_{w,r}$ are the inverse strengths of the shocks s and r respectively. But there is another continuity condition, namely that the streamline direction should be continuous at the contact discontinuity *cd*, so that $\delta_w = \delta_0+\delta_1$, where $\delta_{w,1}$ are the streamline deflexions across w and r. However the Whitham theory has already applied equation (3) to the shock–shock, so the streamline continuity condition cannot be satisfied because the disturbance is already completely determined. The streamline condition has to be abandoned. Now applying equation (8) in turn to all three shocks and substituting the result into equation (14), one obtains, for M_r,

$$M_r^2 = \frac{\gamma-1}{2\gamma}+\frac{((\gamma+1)/2\gamma)\,[M_s^2-(\gamma-1)/2\gamma]}{M_i^2-(\gamma-1)/2\gamma}. \tag{15}$$

Finally the true wave angle ω_1, and the measurable wave angle ω_1' of r (figure 3) follow from

$$\omega_1 = \sin^{-1}(M_r/M_i), \tag{16}$$

$$\omega_1' = \omega_1-\delta_0, \tag{17}$$

and all the quantities ξ_i, θ_w, ω_0, ω_1', χ can be readily measured by experiment.

3. Experiments and results

Although Whitham indicated that his theory would be of value for moderately strong shocks it was decided to compare it also with weak shock data in the hope that it might be of some use for weak MR.

The experiments were done in a conventional shock tube using air, $\gamma = 1{\cdot}402$, in the working section. Plane shock waves were diffracted over a concave corner formed from two rigid steel plates and connected by screw-jacks so that θ_w could be varied continuously. The inverse strength ξ_i was held constant for a given series of experiments, while θ_w was varied from near head-on incidence $\theta_w \to 90°$ to near glancing $\theta_w \to 0°$. There were two values for ξ_i in the weak shock range, one was $\xi_i = 0{\cdot}490$ which was close to the strong shock boundary at $\xi_i = 0{\cdot}433$, and the other was well removed

ξ_i	M_i
$0{\cdot}905 \pm 0{\cdot}005$	$1{\cdot}044 \pm 0{\cdot}002_5$
$0{\cdot}490 \pm 0{\cdot}010$	$1{\cdot}375 \pm 0{\cdot}013$
$0{\cdot}300 \pm 0{\cdot}019_5$	$1{\cdot}732 \pm 0{\cdot}051$
$0{\cdot}150 \pm 0{\cdot}004_5$	$2{\cdot}420 \pm 0{\cdot}035$

TABLE 1. Inverse shock strengths ξ_i and tolerances used in experiments, and their corresponding shock Mach numbers M_i.

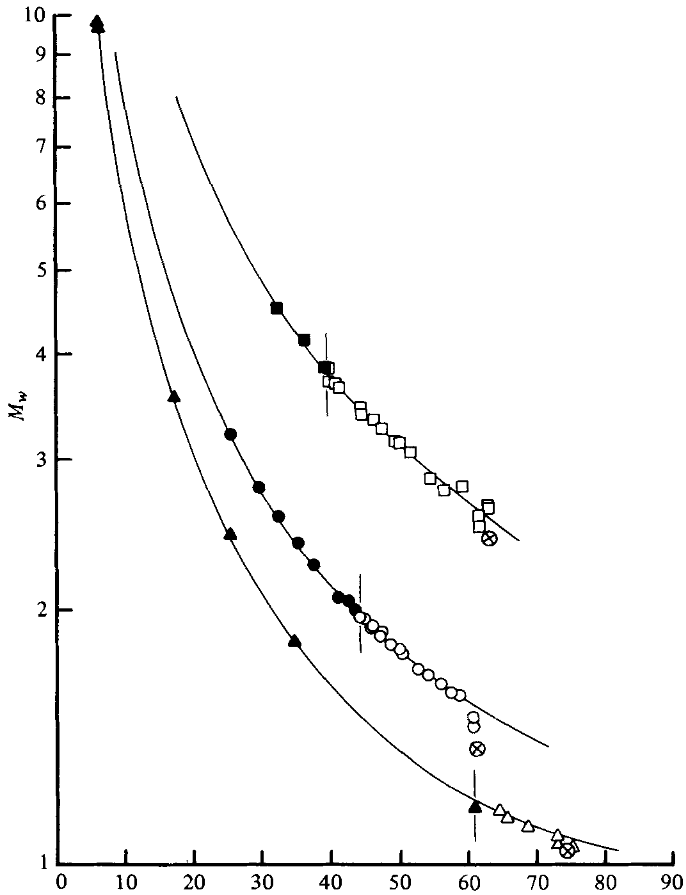

FIGURE 4. Comparison of the Whitham theory with experimental data for the wall Mach number M_w of the Mach stem. □, $\xi_i = 0{\cdot}15$; ○, $\xi_i = 0{\cdot}49$; △, $\xi_i = 0{\cdot}905$; the filled-in symbols are for regular reflexion; the open symbols are for the Mach reflexion. ——, Whitham theory; ⊗, glancing incidence point $\theta_w = 0$ from the exact theory; |, observed transition point between regular and Mach reflexion.

from it at $\xi_i = 0{\cdot}905$. There were also two values in the strong shock range, one, $\xi_i = 0{\cdot}300$, near the boundary and the other, $\xi_i = 0{\cdot}150$, well removed from it. Tolerances were placed on the variations of ξ_i from its nominal values, and data outside these tolerances were rejected, see table 1. The experimental technique has been

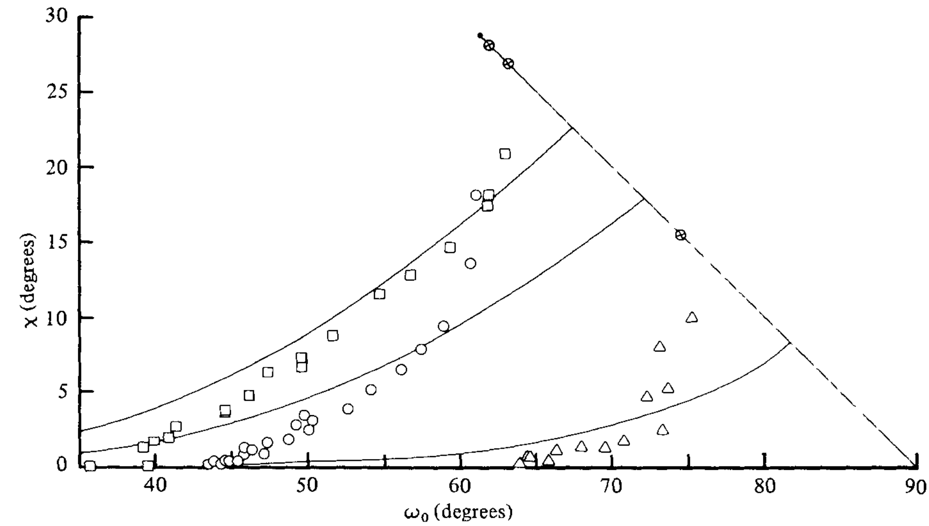

FIGURE 5. Comparison of the Whitham theory with experimental data for the trajectory path angle χ. For an explanation of the symbols see caption to figure 4.

described in detail by Henderson & Siegenthaler (1980). The data obtained for M_w and χ are presented in figures 4 and 5, and that for the measurable wave angle ω_1' in figures 6–9.

4. Discussion

Figure 4 shows that for the M_w data the Whitham theory is in excellent agreement with experiment for both the weak and the strong data. There are signs of some discrepancy near glancing incidence $\theta_w \rightarrow 0$, $\omega_0 \rightarrow \omega_m$, $M_w \rightarrow M_i$. The exact theory of this point has been given by Henderson & Siegenthaler, and it is plotted for each ξ_i in figure 4 with the symbol $\otimes$. The Whitham theory does not pass through this point for any ξ_i, but the experimental data tends towards these points in each case. However, the discrepancy is small, so the theory is accurate even for weak shocks where the corner signal *cs* overtakes the Mach stem *s*. This is a useful result because it means that the Whitham theory for M_w can be combined with the Henderson & Siegenthaler theory to provide a solution for weak Mach reflexion. For regular reflexion the experiments indicate that the Whitham theory behaves like $M_w \rightarrow (M_i/\sin \omega_0)$, that is M_w approaches the free-stream Mach number M_0 of the incident shock *i* along the sloping surface. But M_0 is already known from the given values of $(\gamma, \xi_i, \omega_0)$ and equations (8)–(11) with $\chi = 0$. So the theory tells us nothing new about RR for either weak or strong shocks.

Figure 5 shows that the theory does not agree very well with the data for χ; it is best for $\xi_i = 0{\cdot}15$ but even so there are discrepancies. As we have seen the Whitham theory is in some error near glancing incidence $\theta_w \rightarrow 0$, where now $\chi \rightarrow \chi_m$, and the exact values of χ_m are shown in figure 5. So this accounts for the discrepancy in this region. On the other hand we have seen that the theory always predicts MR even for the RR range of θ_w. Now for RR, $\chi = 0$, and the Whitham theory only has this condition at head-on incidence $\theta_w \rightarrow 90°$, which explains why there is a discrepancy as the diffraction becomes

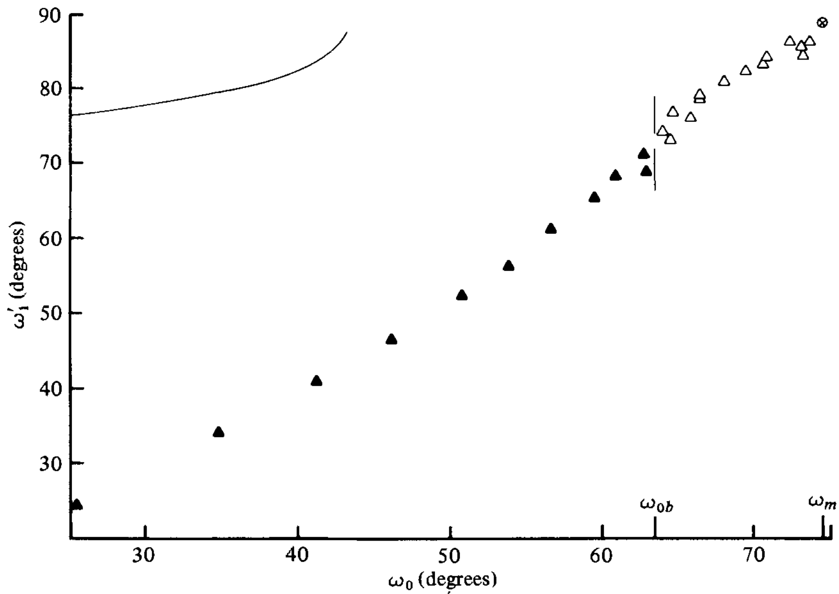

FIGURE 6. Comparison of the Whitham theory with experimental data for the measurable angle of reflexion ω_1', for $\xi_i = 0{\cdot}905 \pm 0{\cdot}005$. ω_{0b}, observed angle of transition between RR and MR; ω_m the value of ω_0 for glancing incidence; for other symbols see caption to figure 4.

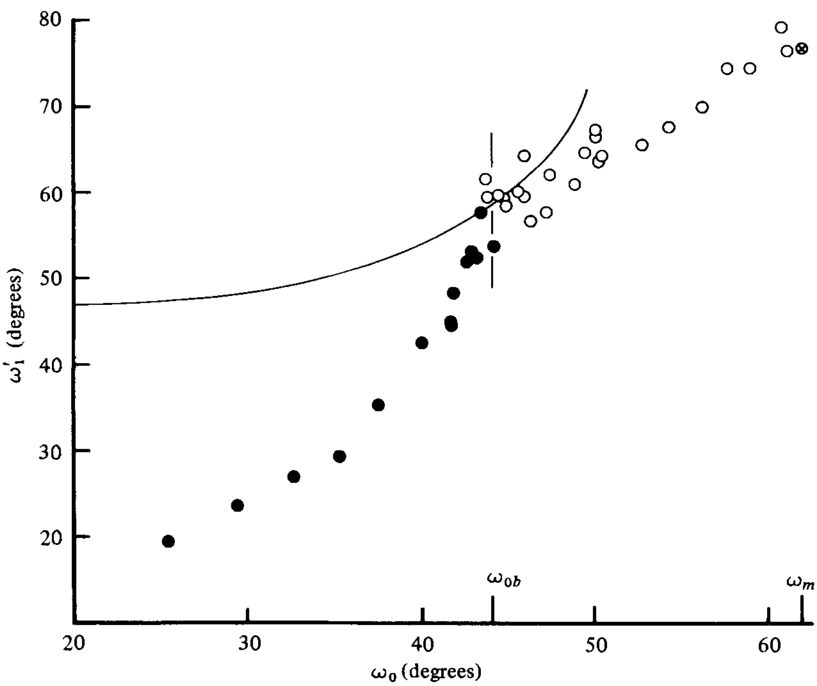

FIGURE 7. Comparison of the Whitham theory with experimental data for the measurable angle of reflexion ω_1', for $\xi_i = 0{\cdot}490 \pm 0{\cdot}010$. For symbols see captions to figures 4 and 6.

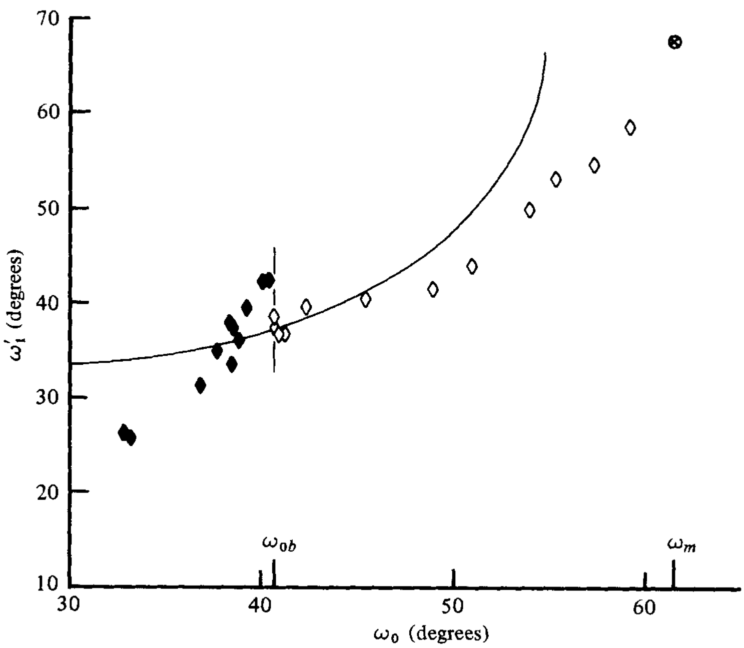

FIGURE 8. Comparison of the Whitham theory with experimental data for the measurable angle of reflexion ω_1' for $\xi_i = 0{\cdot}300 \pm 0{\cdot}019_5$; ◇, $\xi_i = 0{\cdot}300$. For symbols see captions to figures 4 and 6.

regular. Bryson & Gross' (1961) results for a cylinder with $M_i = 2{\cdot}8$ at first sight suggest that the agreement should have been better. However their data is sparse near RR and plotted on such a small scale that it cannot be compared adequately with the present results. Furthermore their shock–shock loci have not been plotted in terms of ω_0 and χ, but in terms of cartesian co-ordinates. The Bryson & Gross results would be consistent with the present results if their values of ω_0 and χ are in the region where the theory crosses the data, as shown in figure 5. The $\xi_i = 0{\cdot}300$ data has been omitted from figures 4 and 5 in the interest of clarity.

For weak shocks the $\xi_i = 0{\cdot}9$ and $0{\cdot}49$ data for ω_1' in figures 6 and 7 show that the shock–shock equations cannot represent the reflected shock r properly. This is to be expected in view of the facts that the corner signal overtakes r and over-determinism does not allow the equation to meet all the constraints on r. The theory improves as the strong shock boundary is approached $\xi_i = 0{\cdot}49 \simeq 0{\cdot}433$, figure 7, but it is still inadequate. The improvement continues as the strong boundary is crossed, $\xi_i = 0{\cdot}300$, figure 8, and here at best the theory would be marginally useful. However for $\xi_i = 0{\cdot}150$ (figure 9) the theory is in excellent agreement with the data for MR except again near glancing incidence. The theory is not applicable to regular reflexion.

5. Conclusions

1. The Whitham theory of shock diffraction accurately determines the shock Mach number of the Mach stem along the sloping wall of a concave corner for both weak and strong Mach reflexion. There is some small error near glancing incidence.

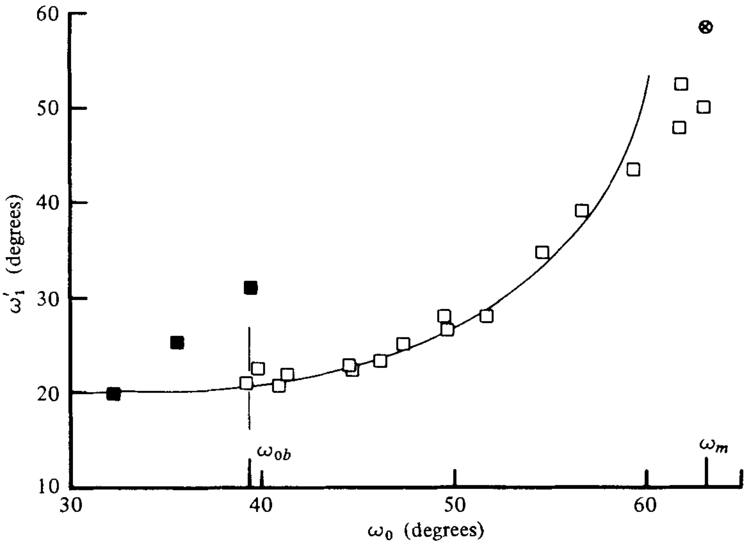

FIGURE 9. Comparison of the Whitham theory with experimental data for the measurable angle of reflexion ω_1' for $\xi_i = 0{\cdot}150 \pm 0{\cdot}035$. For symbols see captions to figures 4 and 6.

2. For strong Mach reflexion the theory can represent the reflected shock adequately except near glancing incidence and near the strong/weak shock boundary, which is at $\xi_i = 0{\cdot}433$ for air. The theory is good for $\xi_i = 0{\cdot}150$ but poor for $\xi_i = 0{\cdot}3$.

3. The theory is not very accurate for determining the trajectory path angle χ of the shock confluence in Mach reflexion.

4. The theory is of no value for regular reflexion.

The author wishes to acknowledge the work of Mr A. Siegenthaler and Mr P. Gray in performing the experiments. This work is supported by the Australian Research Grants Committee.

REFERENCES

Ames Research Staff 1953 Equations, tables and charts for compressible flow. *Nat. Adv. Comm. Aeron., Wash. Rep.* no. 1135.

BAZHENOVA, T. V., GVOZDEVA, L. G. & ZHILIN, YU. V. 1979 *Acta Astron.* **6**, 401.

BLEAKNEY, W. & TAUB, A. H. 1949 *Rev. Mod. Phys.* **21**, 584.

BRYSON, A. E. & GROSS, R. W. F. 1961 *J. Fluid Mech.* **10**, 1.

CHESTER, W. 1954 *Phil. Mag.* **45** (7), 1293.

CHISNELL, R. F. 1957 *J. Fluid Mech.* **2**, 286.

EGGINK, H. 1943 *Zentralstelle Wissenschaftliche Ber., Aachen*, no. 1850.

GUDERLEY, K. G. 1947 *HQ Air Materiel Command, Wright Field, Dayton, Ohio, Tech. Rep.* F-TR-2168 ND.

HENDERSON, L. F. 1964 *Aero. Quart.* **15**, 181.

HENDERSON, L. F. 1965 *Aero. Quart.* **16**, 42.

HENDERSON, L. F. & LOZZI, A. 1975 *J. Fluid Mech.* **68**, 139.

HENDERSON, L. F. & LOZZI, A. 1979 *J. Fluid Mech.* **94**, 541.

HENDERSON, L. F. & SIEGENTHALER, A. 1980 *Proc. Roy. Soc.* A **369**, 537.

KAWAMURA, R. & SAITO, H. 1956 *J. Phys. Soc. Japan* **11**, 584.

MÖLDER, S. 1971 *C.A.S.I. Trans.* **4**, 73.

NEUMANN, J. VON 1963 *Collected Works*, vol. 6. Pergamon.

SAKURAI, A. 1964 *J. Phys. Soc. Japan* **19**, 1440.

SMITH, L. G. 1945 Photographic investigations of the reflexion of plane shocks in air. *O.S.R.D. Rep.* no. 6271.

WECKEN, F. 1949 *Z. angew. Math. Mech.* **29**, 147.

WUEST, W. 1948 *Z. angew. Math. Mech.* **28**, 3.

WHITHAM, G. B. 1957 *J. Fluid Mech.* **2**, 145.

WHITHAM, G. B. 1959 *J. Fluid Mech.* **5**, 369.